

Polyakov relation for the sphere and higher genus surfaces

Pietro Menotti

Dipartimento di Fisica, Università di Pisa

e-mail: pietro.menotti@unipi.it

Abstract

The Polyakov relation, which in the sphere topology gives the changes of the Liouville action under the variation of the position of the sources, in the case of higher genus is related also to the dependence of the action on the moduli of the surface. We write and prove such a relation for genus 1 and for all hyperelliptic surfaces.

1 Introduction

On the sphere topology the Polyakov relation connects the dependence of the action on the position of the sources with the accessory parameters of the related Riemann-Hilbert problem. Such a relation was originally conjectured by Polyakov exploiting the semiclassical limit of quantum operator product expansion [1].

The relation plays a key role in several fields related to Liouville theory like the hamiltonian formulation of $2 + 1$ -dimensional gravity [2, 3, 4] and the study the conformal block expansion of the quantum correlation functions [5, 6, 7, 8]. The accessory parameters appear in the generalized monodromy problem [9, 10] also in connection with the Nekrasov-Shatashvili limit of super Yang-Mills theory [6, 11] and the AGT conjecture [9, 12].

In the simplest case of the sphere topology the Polyakov relation tells us that $\partial S / \partial z_K = -\beta_K / 2$ where S is the on-shell Liouville action, z_K the position of the source and β_K the related accessory parameter.

The proof of Polyakov relation in presence of only parabolic singularities was given in [14] using fuchsian mapping techniques and for the sphere in presence of both parabolic and elliptic singularities using potential theory technique the proof was given in [2, 3] and in [15].

In the present paper we shall extend such kind of relation to higher genus surfaces showing that such a relation takes a different meaning: not only it relates the change of the action under the motion of the sources but also the change of the action under the change of the moduli of the surface. Here we give the proof of the relation in the case of the torus and in the case of all hyperelliptic surface with an arbitrary number of sources.

In the proof of Polyakov relation it is essential to exploit the property of the accessory parameters to be real-analytic functions of the position of the singularities and of the moduli of the surface.

This is not a trivial problem. In paper [16] it was proven that for the sphere the real-analytic dependence of the accessory parameters on the position of the singularities holds everywhere in the restricted case of parabolic and elliptic singularities of finite order; these are the singularities with strength $\eta = (1 - 1/n)/2$.

In ([2, 3]) it was proven for the sphere that the accessory parameters are real-analytic functions of the positions of the singularities and also on the strength η of the singularities in an everywhere dense open set for any collection of elliptic and/or parabolic singularities without the restriction on the elliptic singularities to be of finite order.

For the torus with one source a much stronger result was proven in [17], i.e. that the acces-

sory parameter is a real-analytic function of the coupling and of the modulus everywhere except for a zero measure set.

The proofs of the real-analyticity that we shall give in sections 5,6 rely heavily on the existence and the uniqueness of the solution of the Liouville equation given the strength, the positions of the singularities and the moduli of the surface.

Starting from the papers of Picard [18], which apply only to elliptic singularities, there appeared various proof of the existence and uniqueness of the solution of the Liouville equation [19, 20, 21, 22]. The existence proofs are somewhat lengthy and technical; on the other hand the uniqueness proof is rather straightforward.

The proof of the almost-everywhere real-analytic property of the accessory parameter for the sphere with four sources and for the torus with one source is obtained by applying results and techniques related to analytic varieties [23, 24, 25] even though here we are in presence of a problem of real-analytic varieties [26]. This is dealt with by the techniques of polarization, i.e. by doubling in the intermediate steps of the proof the number of complex variables.

The paper is structured as follows. In section 2 we give the general discussion of the problem. In section 3 we give the action on higher genus surfaces in two different coordinate systems. In section 4 we give the auxiliary differential equation for the torus and all hyperelliptic surfaces with an arbitrary number of sources. In section 5 we give the counting of the degrees of freedom of the parameters appearing in the problem and write the implicit monodromy relations to which the accessory parameters are subject. In section 6 we give a shortened versions of the proof of the real-analyticity property of the accessory parameters in an everywhere dense set in the general case and in the case of the torus or of the four-point function on the sphere, we give a shortened proof of the real-analyticity of the accessory parameter everywhere except for a zero measure set.

In section 7 exploiting the results of the previous sections we give the proof of the Polyakov relation for the sphere, the torus and all hyperelliptic surfaces. In section 8 we give a short discussion of the results of the paper.

In the Appendix using results obtained in [23] we derive the analytic properties of zeros of Weierstrass polynomials which we need in section 6.

2 General discussion

First we outline the semiclassical argument which leads to the Polyakov relation. It will also serve to lay down the notation and fix the normalization of the Liouville action which we shall choose as in [5].

The Liouville action, boundary terms apart, is given by

$$A_L = \frac{1}{\pi} \int (\partial_z \phi \partial_{\bar{z}} \phi + \pi \mu e^{2b\phi}) dz \wedge d\bar{z} \frac{i}{2}. \quad (1)$$

with $z = x + iy$. The holomorphic energy momentum tensor is

$$T_{zz} = T(z) = -(\partial_z \phi)^2 + \mathcal{Q} \partial_z^2 \phi, \quad \mathcal{Q} = \frac{1}{b} + b \quad (2)$$

and for the vertex functions and their dimensions we have

$$V_\alpha(w) = e^{2\alpha\phi(w)}, \quad \Delta_\alpha = \alpha(\mathcal{Q} - \alpha) \quad (3)$$

$$\langle V_{\alpha_1}(w_1) \dots V_{\alpha_n}(w_n) \rangle = \int V_{\alpha_1}(w_1) \dots V_{\alpha_n}(w_n) e^{-A_L[\phi]} D[\phi]. \quad (4)$$

From the operator product expansion we have

$$T(z)V_\alpha(w) = \frac{\Delta_\alpha}{(z-w)^2} V_\alpha(w) + \frac{1}{z-w} \partial_w V_\alpha(w) + \dots \quad (5)$$

To explore the semiclassical limit $b \rightarrow 0$ one sets $\varphi = 2b\phi$ and $\alpha = \frac{\eta}{b}$. The action and the dimension Δ_α become

$$A_L[\phi] = \frac{1}{b^2} S_L[\varphi] \quad \Delta_\alpha \approx \frac{1}{b^2} \eta(1-\eta) \quad (6)$$

where, after performing a constant shift in φ

$$S_L[\varphi] = \frac{1}{2\pi} \int (\frac{1}{2} \partial_z \varphi \partial_{\bar{z}} \varphi + e^\varphi) dz \wedge d\bar{z} \frac{i}{2} \quad (7)$$

and the energy momentum tensor becomes

$$T(z) \approx \frac{1}{b^2} \left[\frac{1}{2} \partial_z^2 \varphi - \frac{1}{4} (\partial_z \varphi)^2 \right] = -\frac{1}{b^2} e^{\frac{\varphi}{2}} \partial_z^2 e^{-\frac{\varphi}{2}}. \quad (8)$$

Then in the semiclassical limit $b \rightarrow 0$

$$\langle V_{\alpha_1}(w_1) \dots V_{\alpha_n}(w_n) \rangle = c e^{-\frac{S^{cl}(w_1, \dots, w_n)}{b^2}} \quad (9)$$

where $S^{cl}(w_1, \dots, w_n)$ is the classical action computed in presence of the sources of strength η_i at the points w_i . As in the semiclassical limit the field is frozen on the classical solution we also have

$$\langle T(z) V_{\alpha_1}(w_1) \dots \rangle = \frac{c}{b^2} \left(\frac{1}{2} \partial_z^2 \varphi(z) - \frac{1}{4} (\partial_z \varphi(z))^2 \right) e^{-\frac{S^{cl}(w_1, \dots, w_n)}{b^2}} \quad (10)$$

where

$$\frac{1}{2} \partial_z^2 \varphi(z) - \frac{1}{4} (\partial_z \varphi(z))^2 = Q(z) = \sum_i \frac{1 - \lambda_i^2}{4(z - w_i)^2} + \frac{\beta_i}{2(z - w_i)}. \quad (11)$$

Comparing with the result obtained using the operator product expansion (5) we have

$$\eta_i(1 - \eta_i) = \frac{1 - \lambda_i^2}{4}, \quad \frac{\partial S^{cl}(w_1, \dots, w_n)}{\partial w_i} = -\frac{\beta_i}{2}. \quad (12)$$

As discussed in the introduction, proofs of (12) have been given in [2, 3, 14, 15] for the topology of the sphere.

In the case of the torus we have two simple representations of the manifold. One is the quotient of the complex z -plane by the group of discrete translations with generators $2\omega_1$, $2\omega_2$ and the other is the Weierstrass representation via the variable $u = \wp(z)$. One can use as parameter classifying the torus the modulus $\tau = \omega_2/\omega_1$ as done in [17], but both for the torus and for higher genus it will be simpler to use the position of the branch points of the two sheet representation of the elliptic or hyperelliptic surface.

For $g = 2$ the analogue of the \wp function was given in [27]. For an approach to the $g = 3$ problem see [28].

On the other hand we know that for any genus $g \geq 2$ we can represent the Riemann surface as the quotient of the z upper half-plane by a fuchsian group i.e. by a standard fundamental curvilinear polygon [29]. Elliptic and hyperelliptic surfaces of any genus can be represented by a two sheet cut u -plane. Even though the transformation between the two representation is not known explicitly except for $g = 1$ and $g = 2$, we find in section 3 general properties of the Jacobian relating the z -representation with the two sheet u -representation of the hyperelliptic surface. This will be sufficient to relate the actions in the two representations.

Accessory parameters appear through the auxiliary ordinary differential equation associated with the Liouville problem. For elliptic and hyperelliptic surfaces they are in number $n + 2g + 1$ for $g \leq 2$, n being the number of sources and g the genus of the surface and for $g \geq 3$ they are in number $3g + n - 1$. However we have relations among them, the fuchsian relations, with the final result that for any g we have $n + 3g - 3$ independent accessory parameters. This is true also in the general case of non-hyperelliptic surfaces [16].

As we mentioned an hyperelliptic surface can be represented in several form. It will turn out that the simplest choice is to use for the moduli the locations u_l of the branch points of the two sheet representation of the manifold; for this choice the Polyakov relation takes the form

$$\frac{\partial S}{\partial u_l} = -B_l - \frac{1}{8}(\partial_s \varphi_M)^2 \quad (13)$$

where B_l is the accessory parameter at the branch point u_l and φ_M is the regular part of the Liouville field at the singularity.

In the process of taking the derivative of the classical action with respect to the locations of the sources u_K or to the moduli u_l , one has to keep in mind that the classical solutions depend on such positions and, through the auxiliary equation, also on the values of the β 's and of a real weight parameter κ which are fixed by the monodromy conditions. Here is where the real-analyticity of the β 's as functions of the u_K, u_l enters the problem.

The normalization of the action S we use in this paper is the one adopted in [5]; it is related to the one used in [17, 30, 31] which we call S_T by $S_T = 2\pi S$ and to the one used in [2, 3] and in [14, 15] which we call S_{CMS} by $S_{CMS} = 4\pi S$.

3 The action on higher genus surfaces

For completeness we start recalling the action on a surface with the topology of the sphere.

$g = 0$.

The sphere is described by $C \cup \infty$ and the action is given by

$$\begin{aligned} S &= \frac{1}{2\pi} \int_{D_\epsilon} \left(\frac{1}{2} \partial\phi \wedge \bar{\partial}\phi + e^\phi dz \wedge \bar{d}z \right) \frac{i}{2} \\ &\quad - \frac{\eta_K}{4\pi i} \oint_{\epsilon_K} \phi \left(\frac{dz}{z - z_K} - \frac{d\bar{z}}{\bar{z} - \bar{z}_K} \right) - \eta_K^2 \log \epsilon_K^2 \\ &\quad + \frac{1}{4\pi i} \oint_R \phi \left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) + \log R^2 \end{aligned} \quad (14)$$

in the limit $\epsilon_K \rightarrow 0$, $R \rightarrow \infty$ where D_ϵ is the disk of radius R in the complex plane from which disks of radius ϵ_K around z_K have been removed. We use the notation $\partial f \equiv \partial_z f dz$, $\bar{\partial} f \equiv \partial_{\bar{z}} f d\bar{z}$. Variation of such an action, with ϕ satisfying at z_K the boundary conditions

$$\phi(z) = -2\eta_K \log(z - z_K)(\bar{z} - \bar{z}_K) + r_K \quad (15)$$

and at $z = \infty$ the boundary condition

$$\phi(z) = -2 \log z \bar{z} + r_\infty \quad (16)$$

where r_K, r_∞ are bounded continuous functions, gives rise to the Liouville equation

$$-\partial_z \partial_{\bar{z}} \phi + e^\phi = 0 \quad (17)$$

in $C \setminus \{u_K\}$. We shall write for the solution of Liouville equation

$$r_K = X_K + o(z - z_K), \quad r_\infty = X_\infty + o\left(\frac{1}{z}\right). \quad (18)$$

The η_K are subject the restrictions $\eta_K < \frac{1}{2}$ (local finiteness of the area) and to the topological restriction $\sum_K 2\eta_K > 2(1 - g) = \chi = 2$, where g is the genus and χ the Euler characteristic.

In the case of parabolic singularities the behavior of the field at the singularities is

$$\phi(z) = -\log(z - z_P)(\bar{z} - \bar{z}_P) - \log(\log(z - z_P)(\bar{z} - \bar{z}_P))^2 + r_P \quad (19)$$

and in the action (14) and in the previous topological relation η_K has to be replaced by $\frac{1}{2}$.

$$g = 1$$

The torus is described by the quotient of the complex plane by the discrete translation group with generators $2\omega_1, 2\omega_2$ and Liouville equation is given by eq.(17) with periodic boundary conditions in z and ϕ behaving as eq.(15,19) at the singularities and $\sum_K 2\eta_K + \sum_P 1 > 2(1 - g) = \chi = 0$.

In such a z -representation the action is given by

$$S_z = \frac{1}{2\pi} \int_T \left(\frac{1}{2} \partial\phi \wedge \bar{\partial}\phi + e^\phi dz \wedge d\bar{z} \right) \frac{i}{2} - \frac{\eta_K}{4\pi i} \oint_{\epsilon_K} \phi \left(\frac{dz}{z - z_K} - \frac{d\bar{z}}{\bar{z} - \bar{z}_K} \right) - \eta_K^2 \log \epsilon_K^2 \quad (20)$$

where the index K runs on the sources. Due to the periodic boundary conditions on ϕ we have no boundary terms. Working with periodic boundary conditions is not very simple. It is useful to go over to the Weierstrass representation of the torus given by the equation

$$w^2 = 4(v - e_1)(v - e_2)(v - e_3), \quad e_1 + e_2 + e_3 = 0. \quad (21)$$

Actually to connect to the general hyperelliptic case it is useful to maintain a more general formalism in which $v = u - (u_1 + u_2 + u_3)/3$ and $e_l = u_l - (u_1 + u_2 + u_3)/3$ so that the equation for the manifold becomes

$$w^2 = 4(u - u_1)(u - u_2)(u - u_3) \quad (22)$$

and

$$\wp(z) = v = u - \frac{u_1 + u_2 + u_3}{3}. \quad (23)$$

From the well known differential equation satisfied by $\wp(z)$

$$(\wp'(z))^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3) \quad (24)$$

we have

$$J = \frac{dz}{du} = \frac{1}{\sqrt{4(u - u_1)(u - u_2)(u - u_3)}}, \quad (25)$$

$$z = \int_{\infty}^u \frac{du}{\sqrt{4(u-u_1)(u-u_2)(u-u_3)}}. \quad (26)$$

A point p of the surface is given by the couple of numbers (u, w) where w satisfies eq.(22) and thus it can assume two values.

For the torus we have for the half-periods

$$\omega_1 = \frac{1}{\sqrt{u_1 - u_2}} K \left(\sqrt{\frac{u_3 - u_2}{u_1 - u_2}} \right) \quad (27)$$

$$\omega_2 = \frac{i}{\sqrt{u_1 - u_2}} K \left(\sqrt{\frac{u_1 - u_3}{u_1 - u_2}} \right) \quad (28)$$

and the modulus is $\tau = \omega_2/\omega_1$. In studying the dependence of the action on the moduli, one can use for the torus τ as done in [17, 30, 31]. On the other hand both for the torus and for the general hyperelliptic surface it is simpler to classify the surfaces in terms of the positions of the branch points of the map from the fundamental standard polygon to the two sheeted u -plane.

Due to the invariance of the area i.e. $e^\varphi du \wedge d\bar{u} = e^\phi dz \wedge d\bar{z}$, in the u -representation the field is given by

$$\varphi(u) = \phi(z) + \log J \bar{J}, \quad J = \frac{dz}{du}. \quad (29)$$

From the behavior of the field $\phi(z)$ at the sources

$$\phi(z) = -2\eta_K \log(z - z_K)(\bar{z} - \bar{z}_K) + X_K + o(z - z_K) \quad (30)$$

we have that the behavior of $\varphi(u)$ at the sources is

$$\varphi(u) = -2\eta_K \log(u - u_K)(\bar{u} - \bar{u}_K) + X_K^u + o(u - u_K) \quad (31)$$

with

$$X_K^u = X_K - (1 - 2\eta_K) \log |4(u_K - u_1)(u_K - u_2)(u_K - u_3)|. \quad (32)$$

At $u = \infty$ being $\phi(0)$ finite we have

$$\varphi(u) = \phi(0) - \log 4 - \frac{3}{2} \log u \bar{u} + o\left(\frac{1}{u}\right). \quad (33)$$

In the following we shall use the convention to denote the dynamical singularities i.e. the sources by u_K with upper case index, while the kinematical singularities describing the Riemann surface in the u -representation will be denoted by u_l , with lower case index.

The action in the u -representation S_u taking into account the behavior (33) is given by

$$\begin{aligned}
S_u &= \frac{1}{2\pi} \int_{D_\varepsilon} \left(\frac{1}{2} \partial\varphi \wedge \bar{\partial}\varphi + e^\varphi du \wedge d\bar{u} \right) \frac{i}{2} \\
&- \frac{\eta_K}{4\pi i} \oint_{\varepsilon_K} \varphi \left(\frac{du}{u - u_K} - \frac{d\bar{u}}{\bar{u} - \bar{u}_K} \right) - \eta_K^2 \log \varepsilon_K^2 \\
&- \frac{1}{16\pi i} \oint_{\varepsilon_l}^d \varphi \left(\frac{du}{u - u_l} - \frac{d\bar{u}}{\bar{u} - \bar{u}_l} \right) - \frac{1}{8} \log \varepsilon_l^2 \\
&+ \frac{1}{8\pi i} \frac{3}{2} \oint_{R_u}^d \varphi \left(\frac{du}{u} - \frac{d\bar{u}}{\bar{u}} \right) + \frac{1}{2} \left(\frac{3}{2} \right)^2 \log R_u^2
\end{aligned} \tag{34}$$

where D_ε is the double sheeted plane and the index d on the contour integrals means that a double turn has to be taken around the kinematical singularities u_l , $l = 1, 2, 3$ and at ∞ in order to come back to the starting point.

For the actions S_z and S_u the general relations [5, 2, 3] hold

$$\frac{\partial S_z}{\partial \eta_K} = -X_K, \quad \frac{\partial S_u}{\partial \eta_K} = -X_K^u \tag{35}$$

which are easily proven from the form (14,20,34) of the actions.

The relation between the two actions is obtained by replacing in S_z , ϕ in terms of φ as given by equation (29). We find

$$S_z = S_u - \sum_K \eta_K (1 - \eta_K) \log [4 |u_K - u_1| |u_K - u_2| |u_K - u_3|] - \frac{1}{2} \log [|u_1 - u_2| |u_2 - u_3| |u_3 - u_1|]. \tag{36}$$

We notice that eq.(36) is consistent with the general relation (35) combined with (32).

The difference between the two actions is of dynamical nature as it involves the source strengths η_K .

In the above equation one recognizes the classical dimensions of the sources $\eta_K(1 - \eta_K)$ multiplied by the logarithm of the Jacobian of the transformation.

$g \geq 2$

We know that a compact Riemann surface of genus $g \geq 2$ can be represented by a standard fundamental domain of the complex upper half-plane. Such a domain is a curvilinear $4g$ -gon which is the analog of the parallelogram T belonging to C which describes the torus. Surfaces of genus $g = 2$ are all hyperelliptic. For these, $g = 2$ Komori [27] gave an explicit representation in terms of the analogue of the Weierstrass function $\wp(z)$, which we shall call $h(z)$, as the ratio of two 6-forms

$$h(z) = \frac{f(z)}{g(z)} \tag{37}$$

where $f(z)$ and $g(z)$ are explicitly written in terms of Poincaré series on a fuchsian group G . Then we have the representation

$$w^2 = 4(u - h(z_1))(u - h(z_2))(u - h(z_3))(u - h(z_4))(u - h(z_5)) \quad (38)$$

with $u = h(z)$.

$$f(z) = \sum_{\gamma \in G} \frac{1}{\gamma z - p_6} P(\gamma z) \gamma'(z)^3 \quad (39)$$

$$g(z) = \sum_{\gamma \in G} P(\gamma z) \gamma'(z)^3 \quad (40)$$

and $f(z)$ has simple poles on the orbit of the point p_6 . $P(z)$ is a properly constructed rational function of z holomorphic in the upper half-plane [27].

In the following we shall enucleate the general features of the transformation between the z and u coordinates for hyperelliptic surfaces of any genus. This will be sufficient to relate S_z with S_u .

The structure of the Jacobian of the transformation

$$J = \frac{dz}{du} \quad (41)$$

can be extracted as follows. The surface is described by

$$w^2 = 4(u - u_1) \dots (u - u_{2g+1}) . \quad (42)$$

(u, w) is a faithful representation of our Riemann surface and thus to each such point there corresponds a point in the standard fundamental polygon in the z -upper-half-plane; z is a locally conformal (analytic invertible) representation of the Riemann surface.

In a domain around a point of M , described by (u, w) with $u \neq u_l$, M is represented by (u, w) with w a determination of $\sqrt{4(u - u_1) \dots (u - u_{2g+1})}$. In a domain around the point of M , described by $(u_l, 0)$, M is faithfully represented by w . In the first case z is an analytic (locally invertible) function of u , while in the second case we have

$$z - z_l = w f_l(w) \quad (43)$$

with f_l analytic and $f_l(0) \neq 0$ and u function of w according to

$$u - u_l = \frac{w^2}{4(u - u_1) \dots \{(u - u_l)\} \dots (u - u_{2g+1})} \quad (44)$$

where the term in $\{\}$ has to be removed.

The Jacobian is given by

$$J = \frac{dz}{du} = \frac{dz}{dw} \frac{dw}{du} = (f_l(w) + w f'_l(w)) \frac{(u - u_1) \dots \{(u - u_l)\} \dots (u - u_{2g+1}) + O(u - u_l)}{\sqrt{(u - u_1) \dots (u - u_{2g+1})}} \\ = 2(f_l(w) + w f'_l(w)) \frac{(u - u_1) \dots \{(u - u_l)\} \dots (u - u_{2g+1}) + O(u - u_l)}{w} \quad (45)$$

so that

$$\begin{aligned} \log J &= -\frac{1}{2} \log(u - u_l) + \log f_l(0) + \frac{1}{2} \log[(u_l - u_1) \dots \{(u_l - u_l)\} \dots (u_l - u_{2g+1})] \\ &+ O(\sqrt{u - u_l}) \\ &\equiv -\frac{1}{2} \log(u - u_l) + j_l + O(\sqrt{u - u_l}) \end{aligned} \quad (46)$$

where

$$j_l = \log f_l(0) + \frac{1}{2} \log[(u_l - u_1) \dots \{(u_l - u_l)\} \dots (u_l - u_{2g+1})] . \quad (47)$$

With regard to the fields we have:

At u_K from

$$\phi(z) = -2\eta_K \log(z - z_K)(\bar{z} - \bar{z}_K) + X_K + o(z - z_K) \quad (48)$$

we deduce

$$\begin{aligned} \varphi(u) &= -2\eta_K \log(u - u_K)(\bar{u} - \bar{u}_K) + (1 - 2\eta_K) \log J_K \bar{J}_K + X_K + O(u - u_K) = \\ &-2\eta_K \log(u - u_K)(\bar{u} - \bar{u}_K) + X_K^u + O(u - u_K) \end{aligned} \quad (49)$$

with

$$X_K^u = X_K + (1 - 2\eta_K) \log J_K \bar{J}_K . \quad (50)$$

At u_l we have

$$\begin{aligned} \varphi(u) &= -\frac{1}{2} \log(u - u_l)(\bar{u} - \bar{u}_l) + j_l + \bar{j}_l + \phi(z_l) + O(\sqrt{u - u_l}) = \\ &-\frac{1}{2} \log(u - u_l)(\bar{u} - \bar{u}_l) + X_l^u + O(\sqrt{u - u_l}) \end{aligned} \quad (51)$$

with

$$X_l^u = j_l + \bar{j}_l + \phi(z_l) . \quad (52)$$

We shall also need the behavior of φ at infinity in u . The local uniformizing variable in the u cut-plane at infinity is v with $v^2 = 1/u$. Then being z in a neighborhood of z_∞ (i.e. of the point which is projected to $u = \infty$) a regular representation of the manifold we have

$$z - z_\infty = \alpha_\infty v + O(v^2) . \quad (53)$$

Thus

$$J = \frac{dz}{du} = \frac{dz}{dv} \frac{dv}{du} = -\frac{\alpha_\infty}{2} u^{-\frac{3}{2}} (1 + O(v)) \quad (54)$$

and

$$\log J = -\frac{3}{2} \log u + j_\infty, \quad j_\infty = \log\left(-\frac{\alpha_\infty}{2}\right). \quad (55)$$

Then as $\varphi = \phi + \log J \bar{J}$ we have

$$\varphi(u) = \phi(z_\infty) - \frac{3}{2} \log u \bar{u} + j_\infty + \bar{j}_\infty = -\frac{3}{2} \log u \bar{u} + X_\infty^u, \quad X_\infty^u = \phi(z_\infty) + j_\infty + \bar{j}_\infty. \quad (56)$$

Integrating

$$\partial_u \partial_{\bar{u}} \varphi = e^\varphi, \quad e^\varphi du \wedge d\bar{u} \frac{i}{2} > 0 \quad (57)$$

we obtain the topological inequality for the source strengths η_K , the number of parabolic singularities and the genus g

$$\begin{aligned} 0 &< \frac{i}{2} \left(- \oint_{u_K} \bar{\partial} \varphi - \oint_{u_l}^d \bar{\partial} \varphi + \oint_\infty^d \bar{\partial} \varphi \right) = \pi \left(\sum_K 2\eta_K - 3 + \sum_P 1 + \sum_{l=1}^{2g+1} 1 \right) \\ &= \pi \left(\sum_K 2\eta_K + \sum_P 1 + 2(g-1) \right). \end{aligned} \quad (58)$$

For $g \geq 2$ the compact Riemann surface is represented by the quotient of the upper z -plane by a Fuchsian group [29]. We refer to a standard fundamental polygon D_z . It is a curvilinear polygon with $4g$ sides lying in the upper z plane with all vertices identified. The sides lie in the order $A_1 B_1 A_1^{-1} B_1^{-1} \dots A_g^{-1} B_g^{-1}$. There exist one and only one element Γ_j^A of the fuchsian group which maps A_j into A_j^{-1} and one and only one element Γ_j^B of the fuchsian group which maps B_j into B_j^{-1} [29].

The side A_j is identified with the side A_j^{-1} and when one runs along the perimeter of the $4g$ -gon the image of A_j , $\Gamma_j A_j = A_j^{-1}$ is is traveled in the opposite direction as A_j . Thus the contour $A_j A_j^{-1}$ is a closed loop on the Riemann surface.

The action in the z -representation is given by

$$\begin{aligned} S_z &= \frac{1}{2\pi} \int_{D_z} \left(\frac{1}{2} \partial \phi \wedge \bar{\partial} \phi + e^\phi dz \wedge d\bar{z} \right) \frac{i}{2} - \frac{\eta_K}{4\pi i} \oint_{\epsilon_K} \phi \left(\frac{dz}{z - z_K} - \frac{d\bar{z}}{\bar{z} - \bar{z}_K} \right) - \eta_K^2 \log \epsilon_K^2 \\ &+ \frac{i}{8\pi} \int_{A_j} \phi (\bar{\partial} \log \bar{s}_j^A - \partial \log s_j^A) + \frac{i}{8\pi} \int_{B_j} \phi (\bar{\partial} \log \bar{s}_j^B - \partial \log s_j^B). \end{aligned} \quad (59)$$

The one dimensional integrals in the last line of the above equation are boundary terms and they are present due to the fact that ϕ is not a scalar but a conformal field, i.e. the periodic boundary conditions are on $e^\phi dz \wedge d\bar{z}$ and not on ϕ . The s are given by $s = dz/dz'$. If the transformation Γ which relates two identified sides A and A^{-1} is given by

$$\Gamma(z) = z' = \frac{az + b}{cz + d} \quad (60)$$

we have $s = (cz + d)^2$ and $\partial \log s = 2c (dz)/(cz + d)$.

In addition D_z excludes small circles of radius ϵ_K around the sources z_K and, as an intermediate step, small circles around z_l , being z_l the images of the u_l and around z_∞ , the image of $u = \infty$. The field dependent boundary terms of the last line in (59) are absent for the torus due to the linear nature of the Γ in such a case.

Substituting in the above equation $\phi = \varphi - \log J \bar{J}$ and using the information on J derived previously in this section, we obtain the relation between the action S_z and the action in the u -representation, S_u

$$\begin{aligned} S_z = & S_u + \eta_K(1 - \eta_K) \log(J_K \bar{J}_K) + \frac{1}{4}(j_l + \bar{j}_l) - \frac{3}{4}(j_\infty + \bar{j}_\infty) \\ & - \frac{i}{8\pi} \int_{A_j} \log(J \bar{J}) (\bar{\partial} \log \bar{s}_j^A - \partial \log s_j^A) - \frac{i}{8\pi} \int_{B_j} \log(J \bar{J}) (\bar{\partial} \log \bar{s}_j^B - \partial \log s_j^B) \\ & - \frac{i}{8\pi} \oint \log(J \bar{J}) \frac{d \log J}{dz} dz \end{aligned} \quad (61)$$

where the last term is the contour integral along the boundary of the standard fundamental domain. Sums over K , l and j are understood.

S_u is given by

$$\begin{aligned} S_u = & \frac{1}{2\pi} \int_{D_u} \left(\frac{1}{2} \partial \varphi \wedge \bar{\partial} \varphi + e^\varphi du \wedge d\bar{u} \right) \frac{i}{2} - \frac{\eta_K}{4\pi i} \oint_{\epsilon_K} \varphi \left(\frac{du}{u - u_K} - \frac{d\bar{u}}{\bar{u} - \bar{u}_K} \right) - \eta_K^2 \log \varepsilon_K^2 \\ & - \frac{1}{16\pi i} \oint_{\epsilon_l}^d \varphi \left(\frac{du}{u - u_l} - \frac{d\bar{u}}{\bar{u} - \bar{u}_l} \right) - \frac{1}{8} \log \varepsilon_l^2 \\ & + \frac{1}{8\pi i} \frac{3}{2} \oint_{R_u}^d \varphi \left(\frac{du}{u} - \frac{d\bar{u}}{\bar{u}} \right) + \frac{1}{2} \left(\frac{3}{2} \right)^2 \log R_u^2 . \end{aligned} \quad (62)$$

Such action with the boundary conditions (49,51,56) is finite. Its variation, again with the boundary conditions (49,51,56), gives rise to the equation of motion

$$- \partial \bar{\partial} \varphi + e^\varphi du \wedge d\bar{u} = 0 \quad (63)$$

in the two-sheeted cut u -plane with the singular points $(u_K, w_K), (u_l, 0)$ removed. The field dependent boundary terms appearing in the last line of eq.(59) are canceled when performing the above described transition from ϕ to φ .

The general relations (35) hold also for the actions (59,62) and they are consistent with the term $\eta_K(1 - \eta_K) \log(J_K \bar{J}_K)$ appearing in eq.(61) and the relation

$$X_K^u - X_K = (1 - 2\eta_K) \log J_K \bar{J}_K . \quad (64)$$

4 The auxiliary differential equation

Given the field $\phi(z)$ we know that in virtue of the Liouville equation

$$e^{\frac{\phi}{2}} \partial_z^2 e^{-\frac{\phi}{2}} \equiv -Q_z(z) \quad (65)$$

is analytic in z except for first and second order poles. Under a change of coordinates e.g. from z to u , the Q transforms as follows

$$Q_u(u) = Q_z(z) \left(\frac{dz}{du} \right)^2 - \{z, u\} \quad (66)$$

where $\{z, u\}$ is the Schwarz derivative

$$\{z, u\} = \left(\frac{dz}{du} \right)^{\frac{1}{2}} \frac{d^2}{du^2} \left(\frac{dz}{du} \right)^{-\frac{1}{2}}. \quad (67)$$

Given the differential equation

$$f''(u) + Q_u(u)f(u) = 0 \quad (68)$$

we know (see e.g. [3, 30, 31]) that the conformal factor can be expressed as

$$e^{\varphi(u)} = \frac{2w_{12}\bar{w}_{12}}{[\kappa^{-2}f_1(u)\bar{f}_1(\bar{u}) - \kappa^2 f_2(u)\bar{f}_2(\bar{u})]^2} \quad (69)$$

where f_1, f_2 are properly chosen solutions of eq.(68) and w_{12} their Wronskian.

The accessory parameters appear in the ordinary differential equation (68) associated with the Liouville problem. We give in the following the structure of the differential equation in canonical form.

For the torus with a single source at $z = z_1$ we have the equation [32]

$$f''(z) + \epsilon(\wp(z - z_1) + \beta)f(z) = 0 \quad (70)$$

but we are interested in the case with n sources and of the general hyperelliptic surface for which the u representation is simpler. The general form of $Q_u(u)$ for any hyperelliptic surface with n sources is

$$\begin{aligned} Q_u &= \frac{3}{16} \left(\frac{1}{(u - u_1)^2} + \cdots + \frac{1}{(u - u_{2g+1})^2} \right) \\ &+ \frac{\beta_1}{2(u - u_1)} + \cdots + \frac{\beta_{2g+1}}{2(u - u_{2g+1})} \\ &+ \sum_K \left(\epsilon_K \frac{(w + w_K)^2}{4(u - u_K)^2 w^2} + \frac{\beta_K(w + w_K)}{4(u - u_K)w} \right) \\ &+ \frac{\beta_w^{(0)}}{w} + \cdots + \frac{u^{g-3} \beta_w^{(g-3)}}{w} \end{aligned} \quad (71)$$

with $\epsilon_K = (1 - \lambda_K^2)/4 = \eta_K(1 - \eta_K)$, $w = \sqrt{4(u - u_1) \dots (u - u_{2g+1})}$ and $w_K = \sqrt{4(u_K - u_1) \dots (u_K - u_{2g+1})}$ where the last line is present only for $g \geq 3$.

The structure of Q_u has the following origin. To each kinematical singularity at u_l with $l = 1 \dots 2g+1$ there corresponds an accessory parameter β_l . To each dynamical singularity at (u_K, w_K) there corresponds an accessory parameter β_K . The factors $(w + w_K)/(2w)$ and their squares project the singularity on the correct sheet. With regard to the kinematical singularities u_l we notice that to the total accessory parameter contribute not only β_l but also the terms $1/w^2$, as explicitly given in section 7.

The function $\frac{1}{w}$ does not introduce a singularity at $w = 0$ as seen going over to the local uniformizing variable $s^2 = u - u_l$ and computing the related Q_s .

At infinity the uniformizing variable is v given by $u = v^{-2}$ and we have for the related Q_v

$$Q_v = Q_u \left(\frac{du}{dv} \right)^2 - \{u, v\}, \quad \frac{du}{dv} = -2v^{-3}, \quad \{u, v\} = \frac{3}{4v^2}. \quad (72)$$

The terms of the last line in eq.(71) are allowed provided, when combined with the other contributions, leave the Q_v free of singularity at $v = 0$. The contribution of the term $\frac{u^p}{w}$ to Q_v is

$$\frac{u^p}{w} \left(\frac{du}{dv} \right)^2 \sim v^{2g-2p-5} \quad \text{for } v \approx 0 \quad (73)$$

and thus they are consistent with the regularity at infinity only for $2g - 2p - 5 \geq 0$ and this happens for $g \geq 3$. The β 's appearing in eq.(71) are subject to the conditions of absence of sources at infinity. The structure is special for $g = 1$ and $g = 2$ while it becomes systematic for $g \geq 3$.

Explicitly:

For $g = 1$ the number β 's is $n + 3$ and we have the three conditions from the regularity at infinity.

$$2(\beta_1 + \beta_2 + \beta_3) + \sum_K \beta_K = 0 \quad (74)$$

$$\frac{3}{2} + \sum_K (\epsilon_K + u_K \beta_K) + 2(u_1 \beta_1 + u_2 \beta_2 + u_3 \beta_3) = 0 \quad (75)$$

$$\sum_K w_K \beta_K = 0 \quad (76)$$

thus leaving n free β 's.

For $g = 2$ the number of β 's is $n + 5$ and we have only two conditions given by

$$2(\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5) + \sum_K \beta_K = 0 \quad (77)$$

$$3 + \sum_K (\epsilon_K + u_K \beta_K) + 2(u_1 \beta_1 + u_2 \beta_2 + u_3 \beta_4 + u_4 \beta_3 + u_5 \beta_5) = 0 \quad (78)$$

leaving us with $n + 3$ β 's.

For $g = 3$ the number of β 's is $8 + n$ and we have the two conditions

$$2(\beta_1 + \cdots + \beta_7) + \sum_K \beta_K = 0 \quad (79)$$

$$\frac{9}{2} + \sum_K (\epsilon_K + u_K \beta_K) + 2(u_1 \beta_1 + \cdots + u_7 \beta_7) = 0 \quad (80)$$

which leaves us with $n + 6$ independent β 's.

From now on, increasing the genus by one we introduce three more β 's while the constraints remain always two. Thus we have recovered from the study of Q_u for the number of independent accessory parameters the general formula $3g - 3 + n$.

From expression (69) we have in a neighborhood of an elliptic singularity u_K with $\zeta = u - u_K$

$$\varphi = -2\eta_K \log(\zeta \bar{\zeta}) - 2 \log [f(\zeta) \bar{f}(\bar{\zeta}) - \kappa^4 (\zeta \bar{\zeta})^{\lambda_K} g(\zeta) \bar{g}(\bar{\zeta})] + 2 \log |\kappa|^2 + \log(2w_{12} \bar{w}_{12}) \quad (81)$$

where $f(\zeta)$ and $g(\zeta)$ are given by a locally convergent power expansions. Around parabolic singularities we have the expression [2, 3]

$$\varphi = -\log \zeta \bar{\zeta} - \log \log^2(\zeta \bar{\zeta}) - 2 \log \left[g(\zeta) \bar{g}(\bar{\zeta}) + \frac{f(\zeta) \bar{g}(\bar{\zeta}) + \bar{f}(\bar{\zeta}) g(\zeta)}{\log \frac{\zeta \bar{\zeta}}{\kappa^4}} \right] + \text{const} . \quad (82)$$

Around a kinematical singularity u_l the local uniformizing variable is s with $s^2 = u - u_l$; in eq.(81) η_K has to be replaced by $1/4$ and f and g become power expansions in s . The detailed form is given in section 7.

At infinity we have for the sphere $\phi = -2 \log z \bar{z} + h(\frac{1}{z}, \frac{1}{\bar{z}})$ and for higher genus $\varphi = -\frac{3}{2} \log u \bar{u} + h(\frac{1}{u}, \frac{1}{\bar{u}})$ with h analytic function in the two variables. This information can be used to give a very simple proof of the uniqueness of the solution of Liouville equation on the sphere, the torus and hyperelliptic surfaces of any genus in presence of any collection of elliptic and parabolic singularities.

Consider two solutions φ_1 and φ_2 of eq.(63) satisfying the above boundary conditions. Then we have

$$\begin{aligned} 0 &\leq \frac{i}{2} \int \partial(\varphi_2 - \varphi_1) \bar{\partial}(\varphi_2 - \varphi_1) = \frac{i}{2} \oint (\varphi_2 - \varphi_1) \bar{\partial}(\varphi_2 - \varphi_1) - \frac{i}{2} \int (\varphi_2 - \varphi_1) \partial \bar{\partial}(\varphi_2 - \varphi_1) \\ &= 0 - \int (\varphi_2 - \varphi_1) (e^{\varphi_2} - e^{\varphi_1}) d^2 u . \end{aligned} \quad (83)$$

The contour integral is around the singularities u_K , u_l and at infinity and due to the behavior of $\varphi_2 - \varphi_1$ it vanishes. Thus we have $\varphi_2 = \varphi_1$. Picard's uniqueness argument

[18] is more complicated because he did not use the information about the non leading terms appearing in eqs.(81,82) provided by the auxiliary differential equation (68).

5 Realization of the $SU(1, 1)$ monodromies

The existence and uniqueness proofs for the solutions of Liouville equations [18, 19, 20, 21, 22] give us information on the accessory parameters. In fact given the solution $\varphi(u)$ we have

$$e^{\phi/2} \partial_u^2 e^{-\phi/2} = -Q(u) \quad (84)$$

The accessory parameters β appear explitletely in the expression of $Q(u)$ (71). Actually on can simply extract each of them by means of a contour integral as written e.g. in [17]. On the other hand if we find a set of accessory parameters and of the real parameter κ such that the monodromies along all cycles and around all singularities are $SU(1, 1)$ then expression (69) provides a single valued solution of Liouville which we know to be unique. The above reasononig shows that we can replace the problem of solving the Liouville equation to the one of finding a set (which we know to be unique) of accessory parameters which make all monodromies $SU(1, 1)$.

In this section we shall write a minimal set the relations which determine the β 's and the κ . All those parameters are necessary to determine the solution. We shall first find a set of relations which are sufficient to determine the β 's and do not involve the κ . Then we give a relation which determines the κ . We remark also that κ intervenes always in the combination $\kappa \bar{\kappa}$ and thus it counts only as one real parameter.

We saw that the number of independent β appearing in Q_u are $n+3g-3$. These correspond to $2n + 6g - 6$ real degrees of freedom. After choosing the elliptic monodromy at u_K for $K = 1$ diagonal, $q_1 = D_1$ we have an additional real degree of freedom given by κ which describes the remnant $SL(2, C)$ transformation.

We have now to use such $2n + 6g - 5$ real degrees of freedom to make all monodromies $SU(1, 1)$. We know from the existence and uniqueness theorem that this can be done and in a unique way.

Here we want to examine how this comes about. For clearness we start from the case of genus $g = 0$ (the sphere).

For $n = 3$ there is no freedom of choice and from the explicit solution in terms of hypergeometric functions (see e.g. [33]) we know that a proper choice of the κ makes q_2 $SU(1, 1)$. Then using $q_1 q_2 q_3 = 1$ we deduce that also that $q_3 \in SU(1, 1)$. For $n = 4$ we have one β

and we can exploit the κ and one real degree of freedom of β to reduce q_2 to the form

$$q_2 = \begin{pmatrix} m_{11} & m_{12} \\ \bar{m}_{12} & m_{22} \end{pmatrix}. \quad (85)$$

We have from the $SL(2, C)$ and the elliptic nature of the transformation

$$m_{11}m_{22} = 1 + m_{12}\bar{m}_{12} \geq 1, \quad m_{11} + m_{22} = -2 \cos \alpha_2 = \text{real}, \quad |2 \cos \alpha_2| \leq 2 \quad (86)$$

which give $m_{22} = \bar{m}_{11}$ i.e. $q_2 \in SU(1, 1)$. We can now use the remaining real degree of freedom to have in q_3 $n_{11} = \rho_1 e^{i\phi}$, $n_{22} = \rho_2 e^{-i\phi}$ and from the reality of the trace we derive $n_{22} = \bar{n}_{11}$. On imposing now

$$\text{tr} D_1 q_2 q_3 = -2 \cos \alpha_4 = \text{real} \quad (87)$$

we obtain for q_3 $n_{21} = \bar{n}_{12}$ i.e. $q_3 \in SU(1, 1)$. Finally due to $q_1 q_2 q_3 q_4 = 1$ we have also $q_4 \in SU(1, 1)$. Increasing n i.e. the number of sources by 1 we gain a further β i.e. two real degrees of freedom and we proceed as above.

For $g > 0$, $n > 0$ we have $n + 2g$ cycles related by the algebraic relation [29]

$$q_1 \dots q_n a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1. \quad (88)$$

After fixing $q_1 = D_1$ diagonal, the $n + 3g - 3$ β 's and κ give us $2n + 6g - 5$ real degrees of freedom to impose the $SU(1, 1)$ nature to the remaining $q_2 \dots q_n a_1 b_1 \dots a_g b_g$. We spend $2(n - 1)$ of them to make all the elliptic $q_2 \dots q_n \in SU(1, 1)$ and we spend the $3(2g - 1)$ remaining degrees of freedom to make $a_1 b_1 \dots a_n$ (but not b_n) $SU(1, 1)$. The 3 degrees of freedom to make each such element $SU(1, 1)$ are spent as follows: two degrees for making $m_{21} = \bar{m}_{12}$ and one for obtaining $|m_{11}| = |m_{22}|$. Then from

$$m_{11}m_{22} = m_{12}m_{21} + 1 \quad (89)$$

we obtain $m_{22} = \bar{m}_{11}$. Using then eq.(88) we reach the equation

$$b_n a_n^{-1} b_n^{-1} \in SU(1, 1) \quad (90)$$

which being already a_n in $SU(1, 1)$, imposes three real constraints on b_n turning it from $SL(2C)$ into $SU(1, 1)$.

A different way to proceed is the following: instead of fixing q_1 diagonal, leave it undetermined. In this way we have in addition to the $2(n + 3g - 3)$ degrees of freedom of the β 's the 3 real degrees of freedom of $SL(2, C)/SU(1, 1)$ in total $2n + 6g - 3$ degrees

of freedom. Spend $2(n-1)$ of them to make the elliptic q_2, \dots, q_n $SU(1,1)$ and $3(2g-1)$ to make $a_1, b_1 \dots a_n$ (but not b_n) $SU(1,1)$. Use the 2 left over parameters to make in b_n $m_{21} = \bar{m}_{12}$ after which b_n assumes the form

$$b_n = \begin{pmatrix} m_{11}/\rho & m_{12} \\ \bar{m}_{12} & \bar{m}_{11}\rho \end{pmatrix} \quad (91)$$

where ρ is a real parameter. The relation

$$\text{tr}(q_2 \dots q_n a_1 b_1 \dots b_g a_g^{-1} b_g^{-1}) = -2 \cos \alpha_1 \quad (92)$$

is a fourth order equation in ρ . From the existence and uniqueness theorem we know that one solution is $\rho = 1$ and such a value makes b_n and thus also $q_1 \in SU(1,1)$, due to eq.(88).

We come now to the writing of the relations which determine the β 's without involving the parameter κ . The relations which solve the monodromies at (u_K, w_K) (dynamical singularities) are given by

$$m_{12}(q_K) = \bar{m}_{21}(q_K) \quad (93)$$

because due to the elliptic nature of the monodromy we have

$$m_{11}(q_K) + m_{22}(q_K) = -2 \cos \alpha_K = \text{real}, \quad |2 \cos \alpha_K| \leq 2 \quad (94)$$

and

$$m_{11}(q_K)m_{22}(q_K) = m_{12}(q_K)\bar{m}_{12}(q_K) + 1 \geq 1. \quad (95)$$

The relations assuring the monodromy along the cycles a_l are

$$m_{12}(a_l) = \bar{m}_{21}(a_l) \quad (96)$$

$$m_{11}(a_l)\bar{m}_{11}(a_l) = m_{22}(a_l)\bar{m}_{22}(a_l) \quad (97)$$

and the same for the b_l with $l = 1 \dots g$.

Below we denote by M_{jk} the monodromy transformation of the two independent solutions f_1, f_2 appearing in eq.(69) along the various cycles. Each matrix element M_{jk} is an analytic functions of the p_j, β_s where $p_j = (u_j, w_j)$ with j running over the n dynamical and the $2g+1$ kinematical singularities and s runs on $3g-3+n$ values. This is the outcome of the solution of the auxiliary differential equation given by the convergent Volterra series. After starting canonical at (u_K, w_K) with $K=1$ we have $q_1 = D_1$ and we spend one degree of freedom to have

$$M_{12}(q_2) = \text{real} \times \bar{M}_{21}(q_2) \quad (98)$$

which can be written as

$$M_{12}(q_2)M_{21}(q_2) = \bar{M}_{12}(q_2)\bar{M}_{21}(q_2) . \quad (99)$$

We spend now $2(n - 2 + 2g - 1)$ real degrees of freedom to impose

$$\frac{M_{12}(q_2)}{\bar{M}_{21}(q_2)} = \frac{M_{12}(x)}{\bar{M}_{21}(x)} \quad (100)$$

where $x = q_3, \dots, q_n, a_1, b_1 \dots a_g$, while $2g - 1$ are needed to have in the $a_1, b_1 \dots a_g$

$$|M_{11}| = |M_{22}| . \quad (101)$$

For satisfying (99), (100) and (101) we need $2n+6g-6$ real parameters which are furnished by the $n + 3g - 3$ complex β 's. Then b_g becomes $SU(1, 1)$ through the relation (88). We notice that κ does not intervene in the above relations and it is determined by

$$\frac{1}{\kappa^2 \bar{\kappa}^2} \frac{M_{12}(q_2)}{\bar{M}_{21}(q_2)} = 1 . \quad (102)$$

The relations (99,100,101) are not pure analytic relations as in all of them the complex conjugate of an analytic function appears. In technical terms it means that the equations (99,100,101) i.e.

$$M_{12}(q_2)M_{21}(q_2) = \bar{M}_{12}(q_2)\bar{M}_{21}(q_2) \quad (103)$$

$$\frac{M_{12}(q_2)}{M_{12}(x)} = \frac{\bar{M}_{21}(q_2)}{\bar{M}_{21}(x)} \quad (104)$$

$$\frac{M_{11}(a_l)}{M_{22}(a_l)} = \frac{\bar{M}_{22}(a_l)}{\bar{M}_{11}(a_l)} \quad l = 1 \dots g, \quad \frac{M_{11}(b_l)}{M_{22}(b_l)} = \frac{\bar{M}_{22}(b_l)}{\bar{M}_{11}(b_l)} \quad l = 1 \dots g - 1 \quad (105)$$

define a real analytic variety [26]. In order to deal with it, it is useful to promote the real variables $\text{Re}\beta_s, \text{Im}\beta_s, \text{Re}u_j, \text{Im}u_j$ to complex variables. An equivalent procedure, which is formally more handy, is the polarization process [24, 26] which consists in considering the variables $\beta_s, \bar{\beta}_s, u_j, \bar{u}_j$ and promoting the $\bar{\beta}_s$ and \bar{u}_j to the independent complex variables β_s^c, u_j^c . Then the results relative to the original problem are obtained for $u_j^c = \bar{u}_j, \beta_s^c = \bar{\beta}_s$. Each equation of the type (104) gives rise to two independent relations of the type

$$\begin{aligned} A(\beta, u) &= \bar{B}(\beta^c, u^c) \\ B(\beta, u) &= \bar{A}(\beta^c, u^c) . \end{aligned} \quad (106)$$

On the other hand relations of the type (103,105) are self-conjugate in the sense that they give rise to the single equation

$$C(\beta, u) = \bar{C}(\beta^c, u^c) . \quad (107)$$

Finally we notice that two self-conjugate relations are equivalent to one “complex” relation e.g.

$$C(\beta, u) = \bar{C}(\beta^c, u^c) \quad D(\beta, u) = \bar{D}(\beta^c, u^c) \quad (108)$$

can be written as

$$F(\beta, u) = \bar{G}(\beta^c, u^c) \quad G(\beta, u) = \bar{F}(\beta^c, u^c) \quad (109)$$

with $F(\beta, u) = C(\beta, u) + iD(\beta, u)$, $G(\beta, u) = C(\beta, u) - iD(\beta, u)$. In this way the eqs.(103, 104,105) can be rewritten as $n + 3g - 3$ pairs of complex relations of the type (106).

6 The real-analyticity of the accessory parameters

In proving Polyakov relation it is necessary to exploit the real analyticity of the dependence of the accessory parameters β and of the parameter κ on the moduli u_K, u_l . Actually due to relation (102) it is sufficient to prove the real-analyticity of the β 's.

On the sphere for any collection of parabolic singularities and of finite order elliptic singularities it was proven by Kra [16] that the accessory parameters are actually real-analytic functions of u_K . Finite order elliptic singularities is the discrete set with source strength $\eta = (1 - 1/n)/2$. For $n \rightarrow \infty$ they accumulate to the parabolic limit.

We are however interested in the case in which the elliptic singularities are arbitrary.

However in this case we have no proof of real-analyticity everywhere and thus our analysis will be of local nature.

A prerequisite in the proof of real-analyticity of the β 's exploiting the monodromy conditions of section 5 is the continuity of the β on the moduli u_K, u_l .

In [3] it was proven using Green function technique that, as expected, the functions $\phi, \partial_u \phi, \partial_u^2 \phi$ are uniformly bounded in any region of the u plane, obtained by excluding finite disks around the singularities, with bounds which depends continuously on u_j . Thus taking contour integrals of eq.(71) at a finite distance from the singularities we have that the β 's are bounded functions of the u_K, u_l when u_K, u_l vary in a small polydisk. Such a result combined with continuity of eqs.(103,104,105) and the uniqueness of the solution implies that the β 's are continuous functions of the u_K, u_l . Continuity is the basic requirement to translate the equations of the previous section into the local analysis of analytic varieties [23].

In the papers [2, 3] for the sphere topology it was proven that the β 's are real-analytic function of the u_K in an everywhere dense open set in the space of the parameters u_K .

For clearness we illustrate the proof in the case of one accessory parameter β , the extension to any number of accessory parameters and moduli being straightforward.

In the following W stays for Weierstrass and WPT for Weierstrass preparation theorem. By Picard solution we understand the unique values $\beta_R(u)$ $\beta_I(u)$ which solves the monodromy problem (also in presence of parabolic singularities). The subscripts R, I stay for the real and imaginary part.

We denote by $\Delta^{(i)}$ the set of relations assuring the $SU(1, 1)$ nature of all monodromies. Given a value u_0 we have $\Delta^{(i)}(\beta_R(u_0), \beta_I(u_0), u_0) = 0$. Let $\Delta^{(1)}(\beta_R, \beta_I(u_0), u_0)$ be non identically zero in β_R . Such $\Delta^{(1)}$ has to exist otherwise we violate the uniqueness result. Then we can apply WPT to translate $\Delta^{(1)}(\beta_R, \beta_I, u_R, u_I) = 0$ into

$$P(\beta_R - \beta_R(u_0)|\beta_I, u_R, u_I) = 0 \quad (110)$$

If P is first order we have

$$\beta_R - \beta_R(u_0) + a_0(\beta_I, u_R, u_I) = 0 \quad (111)$$

and β_R is an analytic function of β_I and u_R, u_I in the W-neighborhood \mathcal{O}_0 of $\beta_I(u_0), u_0$. If P is higher order let us consider P' . If $P'(\beta_R(u_0)|\beta_I(u_0), u_0) = 0$ but not identically zero in a neighborhood of u_0 , then we can solve for β_R for values of u which lie as near as we want to u_0 and these form an open subset \mathcal{O}_1 of \mathcal{O}_0 .

If $P'(\beta_R(u)|\beta_I(u), u) \equiv 0$ in a neighborhood of u_0 then this constitutes a new equation which the Picard solution has to satisfy and we proceed as above. Being the W-polynomial monic the process ends in a finite number of steps and we have the result that β_R is an analytic function $\beta_R(\beta_I, u_R, u_I)$ of u_R, u_I and β_I , for points u laying as near as we want to u_0 and such set \mathcal{O}_1 is an open set.

We now consider $\Delta^{(2)}(\beta_R(\beta_I, u_R, u_I), \beta_I, u_R, u_I) \equiv F(\beta_I, u_R, u_I)$ which is analytic in β_I, u_R, u_I . Such F cannot be independent of β_I otherwise β_I would not be fixed by the monodromy conditions, violating the uniqueness theorem. Then for any point $u \in \mathcal{O}_1$ we compute the W-polynomial

$$P(\beta_I - \beta_I(u_1)|u_R, u_I) = 0 \quad (112)$$

and proceed as above. The result is that β_R, β_I are real-analytic functions of u in an everywhere dense set.

Iterating, the above procedure works also when u is any collection $u_1, u_2 \dots$ of parameters and we have any number of β . In fact the existence and uniqueness result tell us that the $\Delta^{(j)}$ fix completely the solutions.

In the case of a single accessory parameter like the torus with one source or the four point problem on the sphere, a stronger result can be obtained i.e. that the β is a real-analytic function everywhere except for a zero measure set in the u plane [17].

Through polarization [24, 26] i.e. promoting \bar{u} and $\bar{\beta}$ to new independent complex variables u^c, β^c , the single complex equation which imposes the monodromy $A(\beta, u) = \bar{B}(\bar{\beta}, \bar{u})$ is promoted to a system of two equations

$$\begin{aligned} A(\beta, u) &= \bar{B}(\beta^c, u^c) \\ B(\beta, u) &= \bar{A}(\beta^c, u^c) . \end{aligned} \quad (113)$$

At the end we shall be interested only in the self conjugate solutions of the system (113) i.e. those which for $u^c = \bar{u}$ give $\beta^c = \bar{\beta}$. We know such solution to exist and be unique. Applying WPT to the two equations we have

$$P_1(\beta^c - \bar{\beta}(u_0)|\beta, u, u^c) = 0 \quad (114)$$

$$P_2(\beta^c - \bar{\beta}(u_0)|\beta, u, u^c) = 0 . \quad (115)$$

A common solution of the two equations implies the vanishing of the resolvent of the two polynomials

$$R(P_1, P_2) \equiv f(\beta, u, u^c) = 0 . \quad (116)$$

If $f(\beta, u_0, \bar{u}_0)$ vanish identically in β the system has solutions β^c for any choice of β near $\beta(u_0)$ but most important it can be easily proven [17] that we have infinite self-conjugate solutions for $u = u_0, u^c = \bar{u}_0$ and β near $\beta(u_0)$. This violates Picard's uniqueness result. Thus $f(\beta, u, \bar{u})$ has to depend on β and we can apply WPT reducing it to the equation

$$P(\beta - \beta(u_0)|u, u^c) = 0 \quad (117)$$

All the solutions of eq.(117), and in particular the Picard solution, are analytic in u and u^c i.e. real-analytic in u except a zero measure set as shown in the Appendix.

Thus for the case of a single accessory parameter, we have real-analyticity of the Picard solution $\beta(u, \bar{u})$ not only in an everywhere dense open set, but almost everywhere in the space of the moduli.

In the general case of N accessory parameters, as we have shown above, real-analyticity holds in an everywhere dense open set, but we are not aware of a proof of real-analyticity almost everywhere.

7 Derivation of the Polyakov relation

To derive Polyakov relation we shall go over to a finite form for the action S_u i.e. a form which does not contain $\varepsilon \rightarrow 0$ limits as in eq.(62). This is achieved by decomposing the

field φ in a regular and singular part similarly to what originally done in [2, 3]. Starting from

$$e^\varphi = \frac{2w_{12}\bar{w}_{12}}{[\kappa^{-2}f_1\bar{f}_1 - \kappa^2f_2\bar{f}_2]^2} \quad (118)$$

where f_1, f_2 are solutions of

$$f'' + Q_u f = 0 \quad (119)$$

we have near a singularity u_K with $\zeta = u - u_K$

$$f'' + \left(\frac{1 - \lambda_K^2}{4\zeta^2} + \frac{\beta_K}{2\zeta} + \text{regular terms}\right)f = 0 \quad (120)$$

$$f_{1,2} = \zeta^{\frac{1 \mp \lambda_K}{2}} y_{1,2}(\zeta) \quad (121)$$

$$y_1'' + \frac{1 - \lambda_K}{\zeta} y_1' + \left(\frac{\beta_K}{2\zeta} + \text{regular terms}\right)y_1 = 0 \quad (122)$$

$$y_1 = 1 + a\zeta + \dots; \quad a = -\frac{\beta_K}{2(1 - \lambda_K)} = -\frac{\beta_K}{4\eta_K}. \quad (123)$$

Thus

$$\begin{aligned} e^\varphi &= \text{const}(\zeta\bar{\zeta})^{\lambda_K-1}[(1 + a\zeta + \dots)(1 + \bar{a}\bar{\zeta} + \dots) \\ &- \kappa^4(\zeta\bar{\zeta})^{\lambda_K}(1 + b\zeta + \dots)(1 + \bar{b}\bar{\zeta} + \dots)]^{-2} \end{aligned} \quad (124)$$

and then

$$\varphi = \text{const} - (1 - \lambda_K) \log \zeta\bar{\zeta} - 2[a\zeta + \bar{a}\bar{\zeta} + \dots - \kappa^4(\zeta\bar{\zeta})^{\lambda_K}(1 + b\zeta + \bar{b}\bar{\zeta} + \dots)] \quad (125)$$

where $1 - \lambda_K = 2\eta_K$.

Let Ω be a real field which is equal to $-2\eta_K \log(u - u_K)(\bar{u} - \bar{u}_K)$ and to $-\frac{1}{2} \log(u - u_l)(\bar{u} - \bar{u}_l)$ in finite non overlapping disks around the singularities u_K, u_l and equal to $-\frac{3}{2} \log u\bar{u}$ outside a disk of radius R_Ω which includes all singularities. We shall call the union of these regions C . Elsewhere Ω is defined as a smooth field which connects smoothly with the field in the described regions. Notice that Ω depends on the u_K, u_l .

In this way in the decomposition

$$\varphi = \varphi_M + \Omega \quad (126)$$

φ_M is finite and regular both at infinity and at the singularities. Substituting such a decomposition in the action S_u we obtain

$$S_u = \frac{1}{2\pi} \int \left(\frac{1}{2} \partial\varphi_M \wedge \bar{\partial}\varphi_M - \varphi_M \partial\bar{\partial}\Omega - \frac{1}{2} \Omega \partial\bar{\partial}\Omega + e^\varphi du \wedge d\bar{u} \right) \frac{i}{2}. \quad (127)$$

Varying φ_M in (127) we derive the equations of motion for φ_M

$$\partial\bar{\partial}\varphi_M + \partial\bar{\partial}\Omega = e^\varphi du \wedge d\bar{u}. \quad (128)$$

Due to the real-analytic dependence of the β 's and of κ on the parameter u_K the integrand in (127) is continuous in u and u_K and uniformly bounded by an integrable function as u_K varies is a small domain and the derivative of the integrand w.r.t. u_K is continuous and bounded by an integrable function as u_K varies is a small domain. Thus we can take the derivative under the integral symbol

$$\frac{\partial S_u}{\partial u_K} = \frac{i}{4\pi} \int -\varphi_M \partial \bar{\partial} \Omega_K + \Omega_K \partial \bar{\partial} \varphi_M + \frac{1}{2} \Omega_K \partial \bar{\partial} \Omega - \frac{1}{2} \Omega \partial \bar{\partial} \Omega_K \quad (129)$$

where the subscript K stays for $\frac{\partial}{\partial u_K}$. Notice that no $\frac{\partial \varphi_M}{\partial u_K}$ appears due to the equation of motion (128). We notice that in the disk around u_K we have

$$\Omega_K = \frac{2\eta_K}{u - u_K}, \quad \bar{\partial} \Omega_K = 0 \quad (130)$$

while in the remainder of C we have $\Omega_K = 0$ but not necessarily so in the complement of C . The integral in eq.(129) is finite and to perform the integration by parts below it is useful to write it as the limit for ε going to zero of the integral where a disk of radius ε around u_K is excluded. Integrating by parts we have

$$\begin{aligned} \int \Omega_K \partial \bar{\partial} \varphi_M &= - \int d(\Omega_K \partial \varphi_M) + \int \bar{\partial} \Omega_K \wedge \partial \varphi_M \\ &= - \oint_{u_K} \Omega_K \partial \varphi_M - \oint_{u_K} \varphi_M \bar{\partial} \Omega_K + \int \varphi_M \partial \bar{\partial} \Omega_K = - \oint_{u_K} \Omega_K \partial \varphi_M + \int \varphi_M \partial \bar{\partial} \Omega_K \end{aligned} \quad (131)$$

where the last term cancels the first term in eq.(129) and

$$- \frac{i}{4\pi} \oint_{u_K} \Omega_K \partial \varphi_M = - \frac{i}{4\pi} \oint_{u_K} \frac{2\eta_K}{u - u_K} (-2a) du = - \frac{\beta_K}{2}. \quad (132)$$

The minus sign is due the the fact that we are integrating on the boundary of an inner domain. Moreover we have

$$\int \Omega_K \partial \bar{\partial} \Omega = - \oint_{u_K} \Omega_K \partial \Omega + \int \bar{\partial} \Omega_K \wedge \partial \Omega = \int \bar{\partial} \Omega_K \wedge \partial \Omega \quad (133)$$

and

$$- \int \Omega \partial \bar{\partial} \Omega_K = - \oint_{u_K} \Omega \bar{\partial} \Omega_K + \int \partial \Omega \wedge \bar{\partial} \Omega_K = - \int \bar{\partial} \Omega_K \wedge \partial \Omega \quad (134)$$

which cancels (133). Summarizing

$$\frac{\partial S_u}{\partial u_K} = - \frac{\beta_K}{2}. \quad (135)$$

Parabolic singularities are treated in the same way with the same result.

We come now to the variation of the action under the variation of the modulus u_l . The local uniformizing variable around $(u_l, 0)$ is s with $s^2 = u - u_l$. For Q_s we have

$$Q_s = 2B_l + O(s) \quad (136)$$

and

$$B_l = \beta_l + \sum_K \frac{\epsilon_K w_K^2}{8(u_l - u_K)^2(u_l - u_1) \dots \{(u_l - u_l)\} \dots (u_l - u_{2g+1})} \quad (137)$$

is the total accessory parameter at $u = u_l$. The two independent solution of $f'' + Q_s f = 0$ around $s = 0$ are given by

$$f_1 = 1 + a_1 s - B_l s^2 + a_3 s^3 + \dots, \quad f_2 = b_1 s + b_2 s^2 + b_3 s^3 + \dots \quad (138)$$

For the φ we have

$$\begin{aligned} \varphi &= -\frac{1}{2} \log(u - u_l)(\bar{u} - \bar{u}_l) - 2 \left[a_1 s + \bar{a}_1 \bar{s} - \left(B_l + \frac{a_1^2}{2} \right) s^2 - \left(\bar{B}_l + \frac{\bar{a}_1^2}{2} \right) \bar{s}^2 \right. \\ &\quad \left. - \kappa^4 b_1 \bar{b}_1 s \bar{s} + O(s^3) \right] + \text{const} . \end{aligned} \quad (139)$$

Then for the analogue of the integral (132) we have

$$-\frac{i}{4\pi} \oint_{u_l}^d \Omega_l \partial \varphi_M = \frac{i}{4\pi} \oint_0 \frac{1}{s^2} \left(a_1 ds - (2B_l + a_1^2) s ds - \kappa^4 b_1 \bar{b}_1 \bar{s} ds + O(s^2) ds \right) = -B_l - \frac{a_1^2}{2} \quad (140)$$

and thus

$$\frac{\partial S_u}{\partial u_l} = -B_l - \frac{1}{8} (\partial_s \varphi_M)_{s=0}^2 . \quad (141)$$

The factor two of difference between eq.(135) and eq.(141) in the coefficient of B_l is due to the fact that the boundary of a disk around u_l in the u cut-plane is a double turn.

8 Conclusions

Polyakov relation plays an important role in several aspects of Liouville theory like the semiclassical limit of conformal blocks [5, 6, 7, 8], the generalized monodromy problem [9, 10] and the hamiltonian formulation of 2 + 1 dimensional gravity in presence of matter [2, 3, 4].

In this paper we have extended Polyakov relation to all hyperelliptic surfaces with an arbitrary number of sources. For higher genus we have a relation between the accessory parameters and the change of the action induced not only by the change in the position of the sources but also by the change of the moduli.

After imposing the fuchsian conditions the number of independent accessory parameters is $n + 3g - 3$ being n the number of the sources and g the genus of the surface, and they are determined by imposing the monodromy condition around the dynamical singularities and along the fundamental cycles.

In the proof, as it happens already in the simple case of the sphere it is necessary to exploit the real-analyticity of the accessory parameters as functions of the singularities u_K and u_l which represent the position of the sources and the moduli of the surface.

For the case of parabolic and finite order elliptic singularity we know that such real-analyticity property is true everywhere [16]. For a collection of parabolic and arbitrary elliptic singularities we proved that real-analyticity holds in an everywhere dense open set in the space of the parameters u_K, u_l . For the case of the torus with a single source and for the four point case on the sphere we have the stronger result [17] that real-analyticity holds everywhere except for a zero-measure set in the space of the parameters.

Polyakov relation is then simply proven after decomposing the field in a background component, which takes into account the singularities and the behavior at infinity of the Liouville field, and a regular part. With such a decomposition the change of the action reduces to the computation of a single contour integral.

Appendix

In this appendix we derive the analytic properties of the solutions of the equation given by the W-polynomial

$$P(\beta - \beta(u_0)|u, u^c) = (\beta - \beta(u_0))^m + a_{m-1}(u, u^c)(\beta - \beta(u_0))^{m-1} + \dots + a_0(u, u^c) = 0 \quad (142)$$

which appears in section 6. The $a_j(u, u^c)$ are analytic functions of u and u^c with $a_j(u_0, \bar{u}_0) = 0$.

We start by computing the resultant $R(P, P') = f(u, u^c)$ i.e. the discriminant of P . We have two cases:

1. $f(u, \bar{u}) \not\equiv 0$ in the W-neighborhood of u_0, \bar{u}_0 . Then $f(u, u^c)$ can vanish only on a “thin” set [23]. Such a set has zero 4-dimensional Lebesgue measure [25] and the set where $f(u, \bar{u}) = 0$ has zero 2-dimensional Lebesgue measure, as shown at the end of this appendix.

Thus except for such zero measure set we can apply the analytic implicit function theorem to have $\beta(u, u^c)$ analytic function of u, u^c i.e. $\beta(u, \bar{u})$ real analytic function of u .

2. $f(u, \bar{u})$ is identically zero in the W -neighborhood of u_0, \bar{u}_0 . Then by a theorem on polarization [24, 26] we have that $f(u, u^c)$ is identically zero.

In this case we proceed by computing the reduced Gram determinants D_n of the power-vectors of the roots [23]

$$D_n = \begin{vmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_n \\ \cdots & \cdots & \cdots & \cdots \\ s_{n-1} & s_n & \cdots & s_{2n-2} \end{vmatrix} \quad (143)$$

where

$$s_i = \xi_1^i + \xi_2^i + \cdots + \xi_m^i \quad (144)$$

ξ_k being the m roots of P . Being D_n a symmetric polynomial of the roots it is a polynomial in the coefficients $a_k(u, u^c)$ and as such an analytic function of u, u^c . Notice that $D_m = R(P, P')$ [23].

In the present case

$$D_m(u, u^c) \equiv 0 \quad (145)$$

and we compute D_{m-1} . If it is not identically zero it means that the maximum number of distinct roots is $m-1$ and the set where they are $m-1$ is open and given by subtracting from the initial open set the zeros of D_{m-1} which is a thin set and as such of zero measure. In the region where the maximum number of distinct roots is reached all the solutions of (142) (local sheet) are analytic [23], and in particular the Picard solution is analytic.

Suppose now that

$$D_m = D_{m-1} \equiv 0. \quad (146)$$

Then we compute D_{m-2} and proceed as above.

The procedure ends due to the fact that $D_1 \equiv m$. It corresponds to the situation where we have only one m -times degenerate solution i.e.

$$P(\beta - \beta(u_0); u, u^c) = (\beta - \beta(u, u^c))^m = 0 \quad (147)$$

from which we have $\beta(u, u^c) - \beta(u_0) = -\frac{1}{m}a_{m-1}(u, u^c)$ which is analytic in u, u^c and thus $\beta(u, \bar{u})$ real-analytic in u .

Thus the accessory parameter β is an analytic function of u, u^c everywhere except for a thin set. The thin sets in u, u^c have zero 4-dimensional Lebesgue measure [25]. However we are interested in the 2-dimensional measure in the u for $u^c = \bar{u}$, i.e. given a function $f(u, \bar{u})$ analytic in both arguments, we are interested in the measure of the points where it vanishes.

It is simpler to go over to the “real” variables $x = (u + \bar{u})/2$, $y = -i(u - \bar{u})/2$ and write $f(u, \bar{u}) = f_r(x, y)$ which is also analytic in x and y . Given any point (x_0, y_0) it is always possible [24] to perform a real linear invertible change of variables as to make the WPT applicable at that point. Then we can write

$$f_r(x, y) = U(x, y)((x - x_0)^k + a_{k-1}(y)(x - x_0)^{k-1} + \dots a_0(y)) \equiv U(x, y)P(x - x_0|y) \quad (148)$$

with $a_n(y_0) = 0$ and U a unit. The polynomial in (148) for each y can vanish only at a finite number of points (real x). Then denoting with Ξ the function which equals 1 where its argument vanishes and zero otherwise we have

$$\mu = \int dy \int dx \Xi[P(x - x_0|y)] = \int dy 0 = 0 . \quad (149)$$

We can represent the region of the modulus u as the union of a denumerable set of open domains. We have a zero-measure set of possible non real-analyticity points in each domain and the union of such infinite zero measure set has zero measure.

We conclude that β is a real-analytic function of u except for a set of zero 2-dimensional Lebesgue measure in the u plane.

References

- [1] A.M. Polyakov as reported in Refs. [13],[14]
- [2] L. Cantini, P. Menotti and D. Seminara, *Proof of Polyakov conjecture for general elliptic singularities*, Phys. Lett. B 517 (2001) 203, arXiv:hep-th/0105081
- [3] L. Cantini, P. Menotti and D. Seminara, *Liouville theory, accessory parameters and $(2+1)$ -dimensional gravity*, Nucl. Phys. B 638 (2002) 351, arXiv:hep-th/0203103
- [4] L. Cantini, P. Menotti and D. Seminara, *Hamiltonian structure and quantization of $(2+1)$ -dimensional gravity coupled to particles*, Class.Quant.Grav. 18 (2001) 2253, arXiv:hep-th/0011070
- [5] A.B. Zamolodchikov and Al.B. Zamolodchikov, *Structure constants and conformal bootstrap in Liouville field theory*, Nucl. Phys. B 477 (1996) 577, arXiv:hep-th/9506136
- [6] M. Piatek, *Classical torus conformal block, $N=2$ twisted superpotential and the accessory parameter of Lamé equation*, JHEP 1403 (2014) 124, arXiv:1309.7672 [hep-th]

- [7] L. Hadasz and Z. Jaskolski, *Classical Liouville action on the sphere with three hyperbolic singularities*, Nucl.Phys. B694 (2004) 493, arXiv:hep-th/0309267
- [8] L. Hadasz and Z. Jaskolski, *Classical geometry from the quantum Liouville theory*, Nucl.Phys. B724 (2005) 529-554, arXiv:hep-th/0504204
- [9] A. Litvinov, S. Lukyanov, N. Nekrasov and A. Zamolodchikov, *Classical conformal blocks and Painlevé VI*, JHEP 1407 (2014) 144, arXiv:1309.4700 [hep-th]
- [10] N. Nekrasov, A. Rosly, S. Shatashvili, *Darboux coordinates, Yang-Yang functional, and gauge theory*, Nucl.Phys.Proc.Suppl. 216 (2011) 69, arXiv:1103.3919 [hep-th]
- [11] F. Ferrari and M. Piatek, *Liouville theory, $N = 2$ gauge theories and accessory parameters*, JHEP 05 (2012) 025 arXiv:1202.2149 [hep-th]
- [12] L.F. Alday, D. Gaiotto, Y Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, Lett.Math.Phys. 91 (2010) 167, arXiv:0906.3219v2 [hep-th]
- [13] See, e.g., L. Takhtajan, *Topics in quantum geometry of Riemann surfaces: two dimensional quantum gravity*, in: Proc. Internat. School Phys. Enrico Fermi, IOS, Amsterdam, 1996, p. 127; *Semi-classical Liouville theory, complex geometry of moduli spaces, and uniformization of Riemann surfaces* in: New Symmetry Principles in Quantum Field Theory, Cargese, 1991, NATO Adv. Sci. Inst. Ser. B, Vol. 295, Plenum, New York, 1992, p. 383, and references therein.
- [14] P.G. Zograf and L.A. Takhtajan, *On Liouville equation, accessory parameters, and the geometry of Teichmüller space for Riemann surfaces of genus 0*, Math. USSR Sbornik 60 (1988) 143; *On uniformization of Riemann surfaces and the Weyl-Peterson metric on Teichmüller and Schottky spaces*, Math.USSR Sbornik Vol 60 (1988) 297
- [15] L.A. Takhtajan and P.G. Zograf, *Hyperbolic 2-spheres with conical singularities, accessory parameters and Kähler metric on $\mathcal{M}_{0,n}$* , Trans. Am. Math. Soc. 355 (2003) 1857.
- [16] I. Kra, *Accessory parameters for punctured spheres*, Trans. AMS 313 (1989) 589
- [17] P. Menotti, *Accessory parameters for Liouville theory on the torus*, JHEP 12 (2012) 001, arXiv:1207.6884 [hep-th]

- [18] E. Picard, *De l'integration de l'equation $\Delta u = e^u$ sur une surface de Riemann fermée*, Journal de Crelle, 130 (1905) 243
- [19] H. Poincaré, *Les fonctions fuchsienues et l'equation $\Delta u = e^u$* , Comptes rendus de l'Academie de Science, t.126 (1898) 627
- [20] L. Lichtenstein, *Integration der Differentialgleichung $\Delta_2 u = ke^u$ auf geschlossenen Flächen*, Acta mathematica 40 (1915) 1
- [21] R. McOwen, *Conformal metrics in R^2 with prescribed gaussian curvature and positive total curvature*, Indiana University Mathematical Journal, 34 (1985) 97
- [22] M. Troyanov, *Prescribing curvature on compact surfaces with conical singularities*, Transaction of the american mathematical society, 324 (1991) 793
- [23] H. Whitney, *Complex analytic varieties*, Addison Wesley Publ. Co., Reading, Massachusetts Menlo Park, California London Don Mills, Ontario, 1972
- [24] S. Bochner W.T. Martin, *Several Complex variables*, Princeton University Press 1948
- [25] R.C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Inc. 1965
- [26] J. D'Angelo, *Several Complex variables and the geometry of real hypersurfaces*, CRC Press Boca Raton 1993
- [27] Y. Komori, *On the automorphic functions for Fuchsian groups of genus two*, Spaces of Kleinian Groups, London Math.Soc.Lec. Notes 329 (2005) 259 Cambridge University Press.
- [28] J. C. Eilbeck, V. Z. Enolsky, S. Matsutani, Y. Onishi, E. Previato, *Abelian functions for trigonal curves of genus three*, International Mathematics Research Notices, Vol. 2007, Article ID rnm140, arXiv:math/0610019v2 [math.AG]
- [29] H.M. Farkas and I. Kra, *Riemann surfaces*, Springer-Verlag, 1980
- [30] P. Menotti, *Riemann-Hilbert treatment of Liouville theory on the torus*, J. Phys. A 44 (2011) 115403, arXiv:1010.4946 [hep-th]
- [31] P. Menotti, *Riemann-Hilbert treatment of Liouville theory on the torus: The general case*, J. Phys. A 44 (2011) 335401, arXiv:1104.3210 [hep-th]

- [32] L. Keen, H.E. Rauch and A.T. Vasquez, *Moduli of punctured tori and the accessory parameter of Lamé equation*, Trans. Am. Math. Soc. 255 (1979) 201.
- [33] P. Menotti and G. Vajente, *Semiclassical and quantum Liouville theory on the sphere*, Nucl.Phys. B 709 [FS] (2005) 465, arXiv:hep-th/0411003